Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 19, 2003*
Accepted: *December 17, 2003*

Renormalization of hamiltonian field theory; a non-perturbative and non-unitarity approach

Amir H. Rezaeian and Niels R. Walet

Department of Physics, UMIST
PO Box 88, Manchester, M60 1QD, U.K.
E-mail: Rezaeian@Theory.phy.umist.ac.uk, Niels.Walet@umist.ac.uk

Abstract: Renormalization of hamiltonian field theory is usually a rather painful algebraic or numerical exercise. By combining a method based on the many-body coupled cluster method, analyzed in detail by Suzuki and Okamoto, with a wilsonian approach to renormalization, we show that a powerful and elegant method exist to solve such problems. The method is in principle non-perturbative, and is not necessarily unitary.

Keywords: Renormalization Group, Nonperturbative Effects.

Contents

1. Introduction

The power of hamiltonian methods is well known from the study of non-relativistic many-body systems and from strongly-interacting few particle systems, even though a lagrangian approach is usually chosen for relativistic theories. Hamiltonian methods for strongly-interacting systems are intrinsically non-perturbative and usually contain a Tamm-Dancoff type approximation, in the sense that, in practice, one has to limit the bound state as an expansion over states containing a small number of particles. This truncation of the Fock space gives rise to a new class of non-perturbative divergences, since the truncation does not allow us to take into account all diagrams for any given order in perturbation theory. Therefore renormalization issues have to be considered carefully. Two very different remedies for this issue are the use of light-front Tamm-Dancoff field theory (LFFT) [1] and the application of the coupled cluster method (CCM) [2, 3]. In the LFFT the quantization plane is chosen to coincide with the light front, therefore the divergences that plagued the original theories seem to disappear [4]. Furthermore, not having to include interactions in boost operators allows a renormalizable truncation scheme [5]. One of the most important difficulties in LFFT is the complicated structure of the renormalization process [6]. In the standard form of CCM, on the other hand, the amplitudes obey a system of coupled non-linear equations which contain some ill-defined terms because of ultraviolet divergences. It has been shown [7] that the ill-defined amplitudes, which are also called critical topologies, can be systematically removed, by exploiting the linked-cluster property of the ground state. This can be done by introducing a mapping which transfers them into a finite representations without making any approximation such as a coupling expansion. Thus far this resummation method has been restricted to superrenormalizable theories due to its complexity.

A natural question in the renormalization within the hamiltonian formalism arises, since one could also perform renormalization in a lagrangian framework and finally construct the corresponding hamiltonian by means of standard Legendre transformation. (It should be noticed that this is not generally applicable if the lagrangian contains higher-order time-derivatives.) The hamiltonian formalism, despite a certain lack of elegance, has the advantage that it is very economical, and one can use all the know-how of quantum many-body theory.

JHEP12(2003)040

In the last decade extensive attempts have been made to give a prescription for renormalization within the hamiltonian formalism [8, 9, 10]. Commonly unitary transformations are used to decouple the high- and low-energy modes aiming at the partial diagonalization of the hamiltonian. One of the elegant approach in this context has been the so-called similarity renormalization group (SRG) proposed by Glazek and Wilson [8] (and by Wegner [9] independently). The SRG resembles the original wilsonian renormalization group formalism [11], since a transformation that explicitly runs the cutoff is developed. In an early paper, Wilson [11] exploited a similar transformation which was originally introduced by Bloch [12]. There the hamiltonian matrix is reduced by using a transformation which lowers the cutoff initially imposed on the individual states. Later, Wilson abandoned this formalism in favour of a lagrangian one. The most important reason was that the Bloch transformation is ill-defined and produces fake divergences. These divergences emerge from denominators which contain a small energy difference between states retained and states removed by the transformation. The SRG [8, 9] is designed to be free of such small energy denominators and to eliminate interactions which change the energies of the unperturbed states by a large amount. However, there are several issues in this approach: it is hard to incorporate loop expansions within the method, the SRG can not systematically remove interactions which change the number of particles (i.e, when the hamiltonian is not diagonal in particle number space), and most importantly, the computations are complex and there is no an efficient non-perturbative calculable scheme.

In this paper we introduce a method for obtaining the low energy effective operators in the framework of a CCM approach. The transformation constructed avoids the small denominators that plague old-fashioned perturbation theory. Neither perturbation theory nor unitarity of the transformation are essential for this method. The method is non-perturbative, since there is no expansion in the coupling constant; nonetheless, the CCM can be conceived as a topological expansion in the number of correlated excitations. We show that introducing a double similarity transformation using linked-cluster amplitudes will simplify the partial diagonalization underlying renormalization in hamiltonian approaches. However, a price must be paid: due to the truncation the similarity transformations are not unitary, and accordingly the hermiticity of the resultant effective hamiltonian is not manifest. This is related to the fact that we have a biorthogonal representation of the given many-body problem. There is a long tradition of such approaches. The first we are aware of are Dyson-type bosonization schemes [13]. (Here one chooses to map the generators of a Lie algebra, such that the raising generators have a particularly simple representation.) The space of states is mapped onto a larger space where the physically realizable states are obtained by constrained dynamics. This is closely related to CCM formalism, where the extended phase space is a complex manifold, the physical subspace constraint function has been shown to be of second class and the physical shell itself was found to be a Kähler manifold [14]. The second is the Suzuki-Lee method in the nuclear many-body (NMT) problem [15, 16], which reduces the full many-body problem to a problem in a small configuration space and introduces a related effective interaction. The effective interaction is naturally understood as the result of certain transformations which decouple a model space from its complement. As is well know in the theory of effective

JHEP12(2003)040

interactions, unitarity of the transformation used for decoupling or diagonalization is not necessary. Actually, the advantage of a non-unitarity approach is that it can give a very simple description for both diagonalization and the ground state. This has been discussed by many authors [17] and, although it might lead to a non-hermitian effective hamiltonian, it has been shown that hermiticity can be recovered [14, 18]. Nevertheless non-hermiticity is negligible if the model space and its complement are not strongly correlated [19, 20]. Therefore defining a good model space can in principle control the accuracy of CCM.

To solve the relativistic bound state problem one needs to systematically and simultaneously decouple 1) the high-energy from low-energy modes and 2) the many- from the few-particle states. We emphasize in this paper that CCM can in principle be an adequate method to attack both these requirements. Our hope is to fully utilize wilsonian exact renormalization group [22] within the CCM formalism. Here the high energy modes will be integrated out leading to a modified low-energy hamiltonian in an effective many-body space. Notice that our formulation does not depend on the form of dynamics and can be used for any quantization scheme, e.g., equal time or light-cone.

The organization of this paper is as follows. In section 2, we discuss our approach and it's foundation. Finally we conclude and present an outlook in section 3.

2. Formalism

The discussion in this section is partially based on the work of Suzuki and Okamoto [16]. Let us consider a system described by a hamiltonian $H(\Lambda)$ which has, at the very beginning, a large cut-off Λ. We assume that the renormalized hamiltonian $H^{\mathrm{eff}}(\Lambda)$ up to scale Λ can be written as the sum of the canonical hamiltonian and a "counterterm" $H_C(\Lambda)$,

$$H^{\mathrm{eff}}(\Lambda) = H(\Lambda) + H_C(\Lambda)\,. \tag{2.1}$$

Our aim is to construct the renormalized hamiltonian by obtaining this counterterm. Now imagine that we restrict the hamiltonian to a lower energy scale (μ), where we want to find an effective hamiltonian $H^{\mathrm{eff}}(\mu)$ which has the same energy spectrum as the original hamiltonian in the smaller space. Formally, we wish to transform the hamiltonian to a new basis, where the medium-energy modes $\mu < k < \Lambda$, decouple from the low-energy ones, while the low-energy spectrum remains unchanged. We define two subspaces, the intermediate-energy space Q containing modes with $\mu < k < \Lambda$ and a low-energy space P with $\mu \leq k$. Our renormalization approach is based on decoupling of the complement space Q from the model space P. Thereby the decoupling transformation generates a new effective interaction $\delta H(\mu, \Lambda)$ containing the physics between the scale Λ and μ. One can then determine the counterterm by requiring coupling coherence [10, 23], namely that the transformed hamiltonian has the same form as the original one but with Λ every where replaced by μ. (This is in contrast to the popular Effective Field Theory approach, where one includes all permissible couplings of a given order and fixes them by requiring observable computed be both cutoff-independent and Lorentz covariant.) The operators P and Q which project a state onto the model space and its complement, satisfy $P^2 = P$,

JHEP12(2003)040

$Q^2 = Q$, $PQ = 0$ and $P + Q = 1$. We introduce an isometry operator G which maps states in the P- onto the Q- space,

$$|q\rangle = G|p\rangle \qquad (|q\rangle \in Q, |p\rangle \in P)\,. \tag{2.2}$$

The operator G is the basic ingredient in a family of "integrating-out operators", which passes information about the correlations of the high energy modes to the low-energy space. The operator G obeys $G = QGP$, $GQ = 0$, $PG = 0$ and $G^n = 0$ for $n \geqslant 2$. The rather surprising direction for G to act in is due to the definition eq. (2.5) below (cf. the relation between the active and passive view of rotations). In order to give a general form of the effective low-energy hamiltonian, we define another operator $X(n, \mu, \Lambda)$,

$$X(n, \mu, \Lambda) = (1 + G)(1 + G^\dagger G + GG^\dagger)^n\,. \tag{2.3}$$

(n is a real number.) The inverse of $X(n, \mu, \Lambda)$ can be obtained explicitly,

$$X^{-1}(n, \mu, \Lambda) = (1 + G^\dagger G + GG^\dagger)^{-n}(1 - G)\,. \tag{2.4}$$

The special case $n = 0$ is equivalent to the transformation introduced in ref. [24] to relate the hermitian and non-hermitian effective operators in the energy-independent Suzuki-Lee approach. We now consider the transformation of $H(\Lambda)$ defined as

$$\overline{H}(n, \mu, \Lambda) = X^{-1}(n, \mu, \Lambda)H(\Lambda)X(n, \mu, \Lambda)\,, \tag{2.5}$$

where we have

$$H(\Lambda) \to \overline{H}(n, \mu, \Lambda) \equiv H(\mu) + \delta H(\mu, \Lambda)\,. \tag{2.6}$$

One can prove that if $\overline{H}(n, \mu, \Lambda)$ satisfies the desirable decoupling property,

$$Q\overline{H}(n, \mu, \Lambda)P = 0\,, \tag{2.7}$$

or more explicitly, by substituting the definition of $X(n, \mu, \Lambda)$ and $X^{-1}(n, \mu, \Lambda)$ from eqs. (2.3)–(2.4),

$$QH(\Lambda)P + QH(\Lambda)QG - GPH(\Lambda)P - GPH(\Lambda)QG = 0\,, \tag{2.8}$$

that $H^{\text{eff}}(\mu) \equiv \mathcal{H}(n, \mu) = P\overline{H}(n, \mu, \Lambda)P$ is an effective hamiltonian for the low energy degrees of freedom. In other words, it should have the same low-energy eigenvalues as the original hamiltonian. The proof is as follows:

Consider an eigenvalue equation in the P space with $\{|\phi(k)\rangle \in P\}$,

$$P\overline{H}(n, \mu, \Lambda)P|\phi(k)\rangle = E_k P X^{-1}(n, \mu, \Lambda)X(n, \mu, \Lambda)P|\phi(k)\rangle\,. \tag{2.9}$$

By multiplying both sides by $X(n, \mu, \Lambda)$ and making use of the decoupling property eq. (2.7), we obtain

$$H(\Lambda)X(n, \mu, \Lambda)P|\phi(k)\rangle = E_k X(n, \mu, \Lambda)P|\phi(k)\rangle\,. \tag{2.10}$$

JHEP12(2003)040

This equation means that E_k in eq. (2.9) agrees with one of the eigenvalue of $H(\Lambda)$ and $X(n, \mu, \Lambda)P|\phi(k)\rangle$ is the corresponding eigenstate. Now we demand that

$$H^{\text{eff}}(\mu) \equiv H(\mu) + H_C(\mu)\,. \tag{2.11}$$

This requirement uniquely determines the counterterm H_C.

In the same way we can also obtain the Q-space effective hamiltonian, from the definition of $\overline{H}(n, \mu, \Lambda)$. It can be seen that if G satisfies the requirement in eq. (2.8), then we have additional decoupling condition

$$P\overline{H}(n, \mu, \Lambda)Q = 0\,. \tag{2.12}$$

We will argue later that *both* of the decoupling conditions eqs. (2.7) and (2.12) are necessary in order to have a sector-independent renormalization scheme. The word "sector" here means the given truncated Fock space. Let us now clarify the meaning of this concept. To maintain the generality of the previous discussion, we use here the well known Bloch-Feshbach formalism [12, 25, 26]. The Bloch-Feshbach method exploits projection operators in the Hilbert space in order to determine effective operators in some restricted model space. This technique seems to be more universal than Wilson's renormalization formulated in a lagrangian framework. This is due to the fact that in the Bloch-Feshbach formalism, other irrelevant degrees of freedom (such as high angular momentum, spin degrees of freedom, number of particles, etc.) can be systematically eliminated in the same fashion.

Assume that the full space Schrödinger equation is $H(\Lambda)|\psi\rangle = E|\psi\rangle$ and for simplicity $|\psi\rangle$ has been normalized to one. We explicitly construct the energy dependent effective hamiltonian in this formalism,

$$H^{\text{eff}} = P\overline{H}P + P\overline{H}Q\frac{1}{E - Q\overline{H}Q}Q\overline{H}P\,, \tag{2.13}$$

where $\overline{H}$ can be a similarity transformed hamiltonian. This equation resembles the one for Brueckner's reaction matrix (or "G"-matrix) equation in nuclear many-body theory (NMT). In the same way for arbitrary operator O (after a potential similarity transformation), we construct the effective operator

$$\begin{aligned} O^{\text{eff}} &= P\overline{O}P + P\overline{H}Q\frac{1}{E - Q\overline{H}Q}Q\overline{O}P + P\overline{O}Q\frac{1}{E - Q\overline{H}Q}Q\overline{H}P + \\ &\quad + P\overline{H}Q\frac{1}{E - Q\overline{H}Q}Q\overline{O}Q\frac{1}{E - Q\overline{H}Q}Q\overline{H}P\,. \end{aligned} \tag{2.14}$$

The E-dependence in eqs. (2.13) and (2.14) emerges from the fact that the effective interaction in the reduced space is not assumed to be decoupled from the excluded space. However, by using the decoupling conditions introduced in eqs. (2.7) and (2.12), we observe that energy dependence can be removed, and the effective operators become

$$\begin{aligned} H^{\text{eff}} &= P\overline{H}P = \mathcal{H}(n, \mu)\,, \\ O^{\text{eff}} &= P\overline{O}P = \mathcal{O}(n, \mu)\,. \end{aligned} \tag{2.15}$$

The decoupling property makes the operators in one sector independent of the other sector. The effects of the excluded sector is taken into account by imposing the decoupling conditions. This is closely related to the folded diagram method in NMT for removing energy-dependence [27]. (It is well-known in NMT that E-dependence in the G-matrix emerges from non-folded diagrams which can be systematically eliminated using the effective interaction approach). The above argument was given without assuming an explicit form for X and thus the decoupling conditions are more fundamental than the prescription used to derive these conditions. We will show later that one can choose a transformation X, together with the model space and its complement, which avoids the occurrence of "small energy denominators". We now show that Lorentz covariance in a given sector does not hinge on a special form of similarity transformation. We assume ten Poincaré generators L_i satisfying

$$[L_i, L_j] = \sum a_{ij}^k L_k \,, \tag{2.16}$$

where the a_{ij}^k are the known structure coefficients. One can show that if the operators L_i satisfy the decoupling conditions $Q\bar{L}_i P = 0$ and $P\bar{L}_i Q = 0$ then it follows that

$$[L_i^{\text{eff}}, L_j^{\text{eff}}] = \sum a_{ij}^k L_k^{\text{eff}} \,. \tag{2.17}$$

This leads to a relativistic description even after simultaneously integrating out the high-frequency modes and reducing the number of particles. Indeed we conjecture that requiring the decoupling conditions makes the effective hamiltonian free of Lorentz-noninvariant operators for a given truncated sector regardless of the regularization scheme. However, one may still be faced with an effective hamiltonian which violates gauge invariance (for e.g., when sharp cutoff is employed).

Note that the solution to eq. (2.8) is independent of the number n. One can make use of eq. (2.8) and its complex conjugate to show that for any real number n, the following relation for the effective low-energy hamiltonian

$$\mathcal{H}(n,\mu) = \mathcal{H}^\dagger(-n-1,\mu) \,. \tag{2.18}$$

The case $n = -1/2$ is special since the effective hamiltonian is hermitian,

$$\mathcal{H}(-1/2,\mu) = (P + G^\dagger G)^{1/2} H(\Lambda)(P + G)(P + G^\dagger G)^{-1/2} \,. \tag{2.19}$$

Hermiticity can be verified from the relation [28]

$$e^T P = (1 + G)(P + G^\dagger G)^{-1/2} \,, \tag{2.20}$$

where,

$$T = \arctan(G - G^\dagger) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(G(G^\dagger G)^n - \text{h.c.}\right) . \tag{2.21}$$

Since the operator T is anti-hermitian, e^T is a unitary operator. From the above expression eq. (2.19) can be written in the explicitly hermitian form

$$\mathcal{H}\left(-\frac{1}{2},\mu\right) = P e^{-T} H(\Lambda) e^T P \,. \tag{2.22}$$

JHEP12(2003)040

As was already emphasized, renormalization based on unitary transformations is more complicated and non-economical. Thus we will explore a non-unitary approach. An interesting non-hermitian effective low-energy hamiltonian can be obtained for $n = 0$,

$$\mathcal{H}(0, \mu, \Lambda) = PH(\Lambda)(P + QG)\,. \tag{2.23}$$

This form resembles the Bloch and Horowitz type of effective hamiltonian as used in NMT [25], and was used by Wilson in quantum field theory [12]. In the context of the CCM, this form leads to the folded diagram expansion well known in many-body theory [29]. It is of interest that various effective low-energy hamiltonians can be constructed according to eq. (2.15) by the use of the mapping operator G which all obey the decoupling property eq. (2.7) and eq. (2.12). Neither perturbation theory nor hermiticity is essential for this large class of effective hamiltonians.

The CCM approach is, of course, just one of the ways of describing the relevant spectrum by means of non-unitary transformations. According to our prescription the model space is $P : \{|L\rangle \bigotimes |0, b\rangle_h, L \le \mu\}$, where $|0, b\rangle_h$ is a bare high energy vacuum (the ground state of high-momentum of free-hamiltonian) which is annihilated by all the high frequency annihilation operators $\{C_I\}$ (for a given quantization scheme, e.g., equal time or light-cone) , the set of indices $\{I\}$ therefore defines a subsystem, or cluster, within the full system of a given configuration. The actual choice depends upon the particular system under consideration. In general, the operators $\{C_I\}$ will be products (or sums of products) of suitable single-particle operators. We assume that the annihilation and its corresponding creation $\{C_I^\dagger\}$ subalgebras and the state $|0, b\rangle_h$ are cyclic, so that the linear combination of state $\{C_I^\dagger|0, b\rangle_h\}$ and $\{\ {}_h\langle b, 0|C_I\}$ span the decoupled Hilbert space of the high-momentum modes, $\{|H\rangle\}$, where $\mu < H < \Lambda$. It is also convenient, but not necessary, to impose the orthogonality condition, $\langle 0|C_I C_J^\dagger|0\rangle = \delta(I, J)$, where $\delta(I, J)$ is a Kronecker delta symbol. The complement space is $Q : \{|L\rangle \bigotimes (|H\rangle - |0, b\rangle_h)\}$. Our main goal is to decouple the P-space from the Q-space. This gives sense to the partial diagonalization of the high-energy part of the hamiltonian. The states in full Hilbert space are constructed by adding correlated clusters of high-energy modes onto the P-space, or equivalently integrating out the high-energy modes from the hamiltonian,

$$|f\rangle = X(\mu, \Lambda)|p\rangle = e^{\hat{S}} e^{-\hat{S}'}|0, b\rangle_h \bigotimes |L\rangle = e^{\hat{S}}|0, b\rangle_h \bigotimes |L\rangle\,, \tag{2.24}$$

$$\langle \widetilde{f}| = \langle L| \bigotimes\ {}_h\langle b, 0|X^{-1}(\mu, \Lambda) = \langle L| \bigotimes\ {}_h\langle 0|e^{\hat{S}'} e^{-\hat{S}}\,, \tag{2.25}$$

where the operators $X(\mu, \Lambda)$ and $X^{-1}(\mu, \Lambda)$ have been expanded in terms of independent coupled cluster excitations I,

$$\hat{S} = \sum_{m=0} \hat{S}_m \left(\frac{\mu}{\Lambda}\right)^m, \qquad \hat{S}_m = \sum_I{}' \hat{s}_I C_I^\dagger\,,$$
$$\hat{S}' = \sum_{m=0} \hat{S}'_m \left(\frac{\mu}{\Lambda}\right)^m, \qquad \hat{S}'_m = \sum_I{}' \hat{s}'_I C_I\,. \tag{2.26}$$

Here the primed sum means that at least one fast particle is created or destroyed ($I \neq 0$), and momentum conservation in $P \bigoplus Q$ is included in $\hat{s}_I$ and $\hat{s}'_I$. $\hat{S}_m(\hat{S}'_m)$ are not generally commutable in the low-energy Fock space, whereas they are by construction commutable in the high-energy Fock space. It is immediately clear that states in the interacting Hilbert space are normalized, $\langle \widetilde{f}|f\rangle = {}_h\langle b,0|0,b\rangle_h = 1$. We have two types of parameters in this procedure: One is the coupling constant of the theory (λ), and the other is the ratio of cutoffs (μ/Λ). The explicit power counting makes the degree of divergence of each order smaller than the previous one. According to our logic, eq. (2.23) can be written as

$$\mathcal{H}(\mu) = {}_h\langle b,0|X^{-1}(\mu,\Lambda)H(\Lambda)X(\mu,\Lambda)|0,b\rangle_h\,, \tag{2.27}$$

with $X(\mu,\Lambda)$ and $X^{-1}(\mu,\Lambda)$ defined in eq. (2.24) and eq. (2.25). We require that effective hamiltonian $\mathcal{H}(\mu)$ obtained in this way remains form invariant or coherent [10, 23]. This requirement satisfies on an infinitely long renormalization group trajectory and thus does constitutes a renormalized hamiltonian. Thereby one can readily identify the counter terms produced from expansion of eq. (2.27).

It is well-know in many-body applications that the exponential Ansatz eqs. (2.24) and (2.25) guarantees automatically proper size-exclusivity and conformity with the Goldstone linked-cluster theorem to all level of truncation. This parametrization does not manifestly preserve hermitian conjugacy. However, it is compatible with the Hellmann-Feynman theorem (HFT), in other words, demanding hermiticity will violate this theorem at any level of truncation. On the other hand, with this parametrization the phase space $\{\hat{s}_I,\hat{s}'_I\}$ for a given m is a symplectic differentiable manifold. Thereby all the geometrical properties of the configuration space can be precisely defined. (In a more ordinary language, the canonical equations of motion with respect to phase space define a set of trajectories, which fill the whole dynamically allowed region of the phase space.) There is a deep connection between these three properties and we can not give up one without losing something else as well [30]. The individual amplitudes for a given m, $\{\hat{s}^m_I,\hat{s}'^m_I\} \equiv \{\hat{s}_I,\hat{s}'_I\}_m$, can generally be functionals of the low- and high-energy field operators and have to be fixed by the dynamics of quantum system. This is a complicated set of requirements. However, we require less than that. Suppose that after a similar transformation of hamiltonian, $\overline{H}$, we obtain an effective hamiltonian of the form

$$\overline{H} = H(\mathrm{low}) + H_{\mathrm{free}}(\mathrm{high}) + C_I^{\dagger}V_{IJ}C_J\,, \tag{2.28}$$

where V_{IJ} is an arbitrary operator in the low frequency space. The I and J indices should be chosen such that the last term in eq. (2.28) contains at least one creation- operator and one annihilation-operator of high frequency. By using Rayleigh-Schrödinger perturbation theory, it can be shown that the free high-energy vacuum state of $H_{\mathrm{free}}(\mathrm{high})$ is annihilated by eq. (2.28) and remains without correction at any order of perturbation theory. Having said that, we will now consider how to find the individual amplitudes $\{\hat{s}_I,\hat{s}'_I\}_m$ that transfer the hamiltonian into the form eq. (2.28). We split the hamiltonian in five parts:

$$H = H_1 + H_2^{\mathrm{free}}(\mathrm{high}) + V_C(C_I^{\dagger}) + V_A(C_I) + V_B\,, \tag{2.29}$$

where H_1 contains only the low frequency modes with $k \leq \mu$, H_2 is the free hamiltonian for all modes with $\mu < k < \Lambda$, V_C contains low frequency operators and products of the high frequency creation operators $C_I^\dagger$ and V_A is the hermitian conjugate of V_C. The remaining terms are contained in V_B, these terms contain at least one annihilation and creation operators of the high energy modes. Our goal is to eliminate V_C and V_A since V_B annihilates the vacuum. The ket-state coefficients $\{\hat{s}_I\}_m$ are worked out via the ket-state Schrödinger equation $H(\Lambda)|f\rangle = E|f\rangle$ written in the form

$$\langle 0|C_I e^{-\hat{S}} H e^{\hat{S}}|0\rangle = 0\,, \qquad \forall I \neq 0\,. \tag{2.30}$$

The bra-state coefficients $\{\hat{s}_I, \hat{s}'_I\}_m$ are obtained by making use of the Schrödinger equation defined for the bra-state, $\langle\widetilde{f}|H(\Lambda) = \langle\widetilde{f}|E$. First we project both sides on $C_I^\dagger|0\rangle$, then we eliminate E by making use of the ket-state equation projection with the state $\langle 0|e^{\hat{S}'}C_I^\dagger$ to yield the equations

$$\langle 0|e^{\hat{S}'}e^{-\hat{S}}[H, C_I^\dagger]e^{\hat{S}}e^{-\hat{S}'}|0\rangle = 0\,, \qquad \forall I \neq 0\,. \tag{2.31}$$

Alternatively one can in a unified way apply $e^{\hat{S}}e^{-\hat{S}'}C_I^\dagger|0\rangle$ on the Schrödinger equation for the bra-state and obtain

$$\langle 0|e^{\hat{S}'}e^{-\hat{S}}He^{\hat{S}}e^{-\hat{S}'}C_I^\dagger|0\rangle = 0\,, \qquad \forall I \neq 0\,. \tag{2.32}$$

Equation (2.30) and eqs. (2.31) or (2.32) provide two sets of formally exact, microscopic, operatorial coupled non-linear equations for the ket and bra. One can solve the coupled equations in eq. (2.30) to work out $\{\hat{s}_I\}_m$ and then use them as an input in eqs. (2.31) or (2.32).

It is important to notice that eqs. (2.30) and (2.31) can be also derived by requiring that the effective low-energy hamiltonian defined in eq. (2.27), be stationary (i.e. $\delta\mathcal{H}(\mu) = 0$) with respect to all variations in each of the independent functional $\{\hat{s}_I, \hat{s}'_I\}_m$. One can easily verify that the requirements $\delta\mathcal{H}(\mu)/\delta\hat{s}_I = 0$ and $\delta\mathcal{H}(\mu)/\delta\hat{s}'_I = 0$ yield eqs. (2.30) and (2.31). The combination of eqs. (2.30) and (2.31) does not manifestly satisfy the decoupling property as set out in eqs. (2.7) and (2.12). On the other hand eqs. (2.30) and (2.32) satisfy these conditions. Equations (2.30) and (2.32) imply that all interactions including creation and annihilation of fast particles ("I") are eliminated from the transformed hamiltonian $\mathcal{H}(\mu)$ in eq. (2.27). In other words, these are decoupling conditions leading to the elimination of V_C and V_A from eq. (2.29), which is, in essence, a block-diagonalization. Therefore it makes sense for our purpose to use eqs. (2.30) and (2.32) for obtaining the unknown coefficients, losing some of the elegance of the CCM elsewhere.

So far everything has been introduced rigorously without invoking any approximation. In practice one needs to truncate both sets of coefficients $\{\hat{s}_I, \hat{s}'_I\}_m$ at a given order of m. A consistent truncation scheme is the so-called SUB(n,m) scheme, where the n-body partition of the operator $\{\hat{S}, \hat{S}'\}$ is truncated so that one sets the higher partition with $I > n$ to zero at a given accuracy m. Notice that, eqs. (2.31) and (2.32) provide two equivalent sets of equations in the exact form, however after the truncation they can in principle be different. Eqs. (2.30) and (2.32) are compatible with the decoupling property at any level of the truncation, whereas eqs. (2.30) combined with (2.31) are fully consistent with HFT at any level of truncation. Thus the low-energy effective form of an arbitrary operator can be computed according to eq. (2.15) in the same truncation scheme used for the renormalization of the hamiltonian. In particular, we will show that only in the lowest order ($m = 0$), equations (2.31) and (2.32) are equivalent, independent of the physical system and the truncation scheme.

Although our method is non-perturbative, perturbation theory can be recovered from it. In this way, its simple structure for loop expansion will be obvious and we will observe that at lower order hermiticity is preserved. Now we illustrate how this is realizable in our approach. Assume that V_C and V_A are of order λ, we will diagonalize the hamiltonian, at leading order in λ up to the desired accuracy in μ/Λ. We use the commutator-expansion

$$e^{-S}He^{S} = H + [H,S] + \frac{1}{2!}[[H,S],S] + \cdots . \tag{2.33}$$

Eq. (2.30) can be organized perturbatively in order of m, aiming at elimination of the high momenta degree of freedom up to the first order in the coupling constant, thus yields

$$\begin{aligned}
m=0:\ & \langle 0|C_I(V_C + [H_2,\hat{S}_0])|0\rangle = 0\,,\\
m=1:\ & \langle 0|C_I([H_1,\hat{S}_0] + [H_2,\hat{S}_1] + [V_A,\hat{S}_1] + [V_C,\hat{S}_1])|0\rangle = 0\,,\\
& \vdots\\
m=n:\ & \langle 0|C_I([H_1,\hat{S}_{n-1}] + [H_2,\hat{S}_n] + [V_A,\hat{S}_n] + [V_C,\hat{S}_n])|0\rangle = 0\,,
\end{aligned} \tag{2.34}$$

where $I \neq 0$. Notice that $\hat{S}_0$ is chosen to cancel V_C in the effective hamiltonian, hence it is at least of order of λ, consequently it generates a new term $[H_1,\hat{S}_0]$ which is of higher order in μ/Λ and can be canceled out on the next orders by $\hat{S}_1$. The logic for obtaining the equations above is based on the fact that $\hat{S}_n$ should be smaller than $\hat{S}_{n-1}$ (for sake of convergence) and that the equations should be consistent with each other. Since $H_2, V_A, V_C \approx \Lambda$ and $H_1 \approx \mu$, from eq. (2.34) we have the desired relation $\hat{S}_n \approx \frac{\mu}{\Lambda}\hat{S}_{n-1}$. The same procedure can be applied for eq. (2.32) which leads to the introduction of a new series of equations in order of m,

$$\begin{aligned}
m=0:\ & \langle 0|(V_A - [H_2,\hat{S}'_0])C_I^\dagger|0\rangle = 0\,,\\
m=1:\ & \langle 0|([H_1,\hat{S}'_0] + [H_2,\hat{S}'_1] + [V_C,\hat{S}'_1] + [V_A,\hat{S}'_1] - [V_A,\hat{S}_1])C_I^\dagger|0\rangle = 0\,,\\
& \vdots\\
m=n:\ & \langle 0|([H_1,\hat{S}'_{n-1}] + [H_2,\hat{S}'_n] + [V_C,\hat{S}'_n] + [V_A,\hat{S}'_n] - [V_A,\hat{S}_n])C_I^\dagger|0\rangle = 0\,.
\end{aligned} \tag{2.35}$$

Alternatively, we can use eq. (2.31) to yield the equations

$$\begin{aligned}
m=0:\ & \langle 0|\big([V_A,C_I^\dagger] - \big[[H_2,C_I^\dagger],\hat{S}'_0\big]\big)|0\rangle = 0\,,\\
m=1:\ & \langle 0|\big(\big[[V_A,C_I^\dagger],\hat{S}_1\big] - \big[[H_2,C_I^\dagger],\hat{S}'_1\big] - \big[[V_A,C_I^\dagger],\hat{S}'_1\big]\big)|0\rangle = 0\,,\\
& \vdots\\
m=n:\ & \langle 0|\big(\big[[V_A,C_I^\dagger],\hat{S}_n\big] - \big[[H_2,C_I^\dagger],\hat{S}'_n\big] - \big[[V_A,C_I^\dagger],\hat{S}'_n\big]\big)|0\rangle = 0\,.
\end{aligned} \tag{2.36}$$

It is obvious that at order $m = 0$, eqs. (2.35) and (2.36) are the same and $\hat{S}'_0 = \hat{S}_0^\dagger$, which indicates that the similarity transformation at this level remains unitary. It should be noted that diagonalization at first order in the coupling constant introduces a low-energy effective hamiltonian in eq. (2.27) which is valid up to the order λ^3. In the same way, diagonalization at second order in λ modifies the hamiltonian at order λ^4 and leads generally to a non-unitarity transformation. In this way one can proceed to diagonalize the hamiltonian at a given order in λ with desired accuracy in μ/Λ . Finally, the renormalization process is completed by introducing the correct $Z(\Lambda)$ factors which redefine the divergences emerging from eq. (2.27).

3. Conclusion and outlook

In this paper we have outlined a strategy to derive effective renormalized operators in the hamiltonian formulation of a field theory. An example of the application of the method can be found in ref. [21]. The effective low-frequency operator is obtained by the condition that it should exhibit decoupling between the low- and high-frequency degrees of freedom. We showed that the similarity transformation approach to renormalization can be systematically classified. The non-hermitian formulation gives a very simple description of decoupling, leading to a partial diagonalization of the high-energy part. In the recent paper [21], we showed that non-unitarity representation inherent in our formulation is in favour of economic computation and does not produce any non-hermiticity in the relevant terms. The techniques proposed are known from the coupled cluster many-body theory and invoke neither perturbation nor unitarity transformation. We showed that our formalism can be solved perturbatively. In this way, it was revealed that diagonalization at first order in coupling constant defines a correct low-energy effective hamiltonian which is valid up to the order λ^3. One can show that the non-hermiticity of the effective hamiltonian is controllable and might appear in higher order which is beyond our approximation or in irrelevant terms which can be ignored in renormalization group sense.

One of the key features which has not yet been exploited is the non-perturbative aspect of the method; it may well be able to obtain effective degrees of freedom that are very different from the ones occurs at the high-energy scale. This is a promising avenue for future work.

Acknowledgments

One of the authors (A.H.R.) acknowledges support from British Government ORS award and UMIST grant. The work of N.R.W. is supported by the UK engineering and physical sciences research council under grant GR/N15672.

References

[1] R.J. Perry, A. Harindranath and K.G. Wilson, *Light-front Tamm-Dancoff field theory*, *Phys. Rev. Lett.* **65** (1990) 2959.

[2] R.F. Bishop *The coupled cluster method*, in *Microscopic quantum many body theories and their applications*, J. Navarro and A. Polls eds., *Lecture Notes in Physics* **510** (1997) 1, and references therein.

[3] For example G. Hasberg and H. Kümmel, *Coupled cluster description of pion - nucleon systems*, *Phys. Rev.* **D 33** (1986) 1367;
H.G. Kümmel, *Higher order coupled cluster approximation to the vacuum of* Φ^4 *field theory in (1+1)-dimensions*, *Phys. Rev.* **D 50** (1994) 6556.

[4] J.P. Vary and A. Harindranath, *In nuclear and particle physics on the light cone*, M.B. Johnson and L.S. Kisslinger eds., World Scientific, Singapore 1989, and references therein.

[5] S. Weinberg, *Dynamics at infinite momentum*, *Phys. Rev.* **150** (1966) 1313.

[6] For example R.J. Perry and A. Harindranath, *Renormalization in the light front Tamm-Dancoff approach to field theory*, *Phys. Rev.* **D 43** (1991) 4051;
K.G. Wilson, T.S. Walhout, A. Harindranath, W.M. Zhang, R.J. Perry and S.D. Glazek, *Nonperturbative QCD: a weak coupling treatment on the light front*, *Phys. Rev.* **D 49** (1994) 6720 [hep-th/9401153];
M.M. Brisudova and R. Perry, *Initial bound state studies in light-front QCD*, *Phys. Rev.* **D 54** (1996) 1831 [hep-ph/9511443].

[7] U. Kaulfuss, *Renormalization of the coupled cluster equations in three-dimensional* Φ^4 *quantum field theory*, *Phys. Rev.* **D 32** (1985) 1421.

[8] S.D. Glazek and K.G. Wilson, *Renormalization of hamiltonians*, *Phys. Rev.* **D 48** (1993) 5863; *Perturbative renormalization group for hamiltonians*, *Phys. Rev.* **D 49** (1994) 4218.

[9] F. Wegner, *Flow-equations for hamiltonians*, *Ann. Physik (Berlin)* 3 (1994) 77.

[10] For example A.P. Szczepaniak and E.S. Swanson, *From current to constituent quarks: a renormalization group improved hamiltonian-based description of hadrons*, *Phys. Rev.* **D 55** (1997) 1578 [hep-ph/9609525];
E.L. Gubankova and F. Wegner, *Flow equations for qed in the light front dynamics*, *Phys. Rev.* **D 58** (1998) 025012 [hep-th/9710233];
T. Maslowski and M. Wieckowski, *4th order similarity renormalization of a model hamiltonian*, *Phys. Rev.* **D 57** (1998) 4976 [hep-th/9707057];
T.S. Walhout, *Similarity renormalization, hamiltonian flow equations and Dyson's intermediate representation*, *Phys. Rev.* **D 59** (1999) 065009 [hep-th/9806097];
D.G. Robertson, E.S. Swanson, A.P. Szczepaniak, C.R. Ji and S.R. Cotanch, *Renormalized effective QCD hamiltonian: gluonic sector*, *Phys. Rev.* **D 59** (1999) 074019 [hep-ph/9811224];
G. Alexanian and E.F. Moreno, *On the renormalization of hamiltonians*, *Phys. Lett.* **B 450** (1999) 149 [hep-th/9811158];
D. Chakrabarti and A. Harindranath, *Mesons in (2+1) dimensional light front QCD, II: similarity renormalization approach*, *Phys. Rev.* **D 65** (2002) 045001 [hep-th/0110156].

[11] K.G. Wilson, *Model hamiltonians for local quantum field theory*, *Phys. Rev.* **B 140** (1965) 445; *A model of coupling constant renormalization*, *Phys. Rev.* **D 2** (1970) 1438.

[12] C. Bloch, *Sur la théorie des perturbations des états liés*, *Nucl. Phys.* 6 (1958) 329.

[13] J.S. Arponen, R. F. Bishop and E. Pajanne, *Extended coupled-cluster method, I. Generalized coherent bosonization as a mapping of quantum theory into classical hamiltonian mechanics*, *Phys. Rev.* **A 36** (1987) 2519.

[14] J. Arponen, *Constrained hamiltonian approach to the phase space of the coupled cluster method*, *Phys. Rev.* **A 55** (1997) 2686.

[15] K. Suzuki and S. Y. Lee, *Convergent theory for effective interaction in nuclei*, *Prog. Theor. Phys.* **64** (1980) 209.

[16] K. Suzuki and R. Okamoto, *Unitary-model-operator approach to nuclear many-body problem, I. the ground-state and one-body energies of 16O*, *Prog. Theor. Phys.* **75** (1986) 1388; *Unitary-model-operator approach to nuclear many-body problem, II. Three-body-cluster effects on properties of 16O Prog. Theor. Phys.* **76** (1986) 127; *Effective interaction theory and unitary-model-operator approach to nuclear saturation problem*, *Prog. Theor. Phys.* **92** (1994) 1045.

[17] K. Suzuki, *Reformulation of coupled-cluster theory for many-fermion system on similarity-transformation theory*, *Prog. Theor. Phys.* **87** (1992) 937;
H. Kümmel, *Effective operators in the relativistic meson nucleon system*, *Phys. Rev.* **D 27** (1983) 765;
K. Suzuki, R. Okamoto and H. Kumaga, *Many-body theory in terms of effective interactions and its relation to coupled-cluster method*, *Nucl. Phys.* **A 580** (1994) 213 and references therein.

[18] K. Suzuki,*Construction of hermitean effective interaction in nuclei. General relation between hermitean and non-hermitean forms*, *Prog. Theor. Phys.* **68** (1982) 246;
T.T.S. Kuo, P.J. Ellis, J. Hao, Z. Li, K. Suzuki, R. Okamoto and H. Kumagai, *Hermitian effective interactions derived from the paris and bonn potentials*, *Nucl. Phys.* **A 560** (1993) 621.

[19] K. Suzuki, R. Okamoto, P.J. Ellis, J. Hao, Z. Li, T.T.S. Kuo, *Origin of the nonhermiticity of effective interactions*, *Phys. Lett.* **B 308** (1993) 1.

[20] C.-S. Hsue, H. Kümmel and P. Uberholz, *Coupled cluster description of field theories: procedures and their application to the vacuum sector in (1+1)-dimensional Φ^4 field theories*, *Phys. Rev.* **D 32** (1985) 1435.

[21] A.H. Rezaeian and N.R. Walet, *Linked-cluster Tamm-Dancoff field theory*, *Phys. Lett.* **B 570** (2003) 129 [hep-th/0306159].

[22] K.G. Wilson and J.B. Kogut, *The renormalization group and the epsilon expansion*, *Phys. Rept.* **12C** (1974) 75;
K.G. Wilson, *The renormalization group: critical phenomena and the kondo problem*, *Rev. Mod. Phys.* **47** (1975) 773;
C. Bagnuls and C. Bervillier, *Exact renormalization group equations: an introductory review*, *Phys. Rept.* **348** (2001) 91 [hep-th/0002034].

[23] R.J. Perry and K.G. Wilson, *Perturbative renormalizability with an infinite number of relevant and marginal operators*, *Nucl. Phys.* **B 403** (1993) 587;
R. Oehme, K. Sibold and W. Zimmermann, *Renormalization group equations with vanishing lowest order of the primary beta function*, *Phys. Lett.* **B 147** (1984) 115.

[24] D. Navratil, H.B. Geyer and T.T.S. Kuo, *Energy independent hermitian and non-hermitian effective operators*, *Phys. Lett.* **B 315** (1993) 1.

[25] C. Bloch and J. Horowitz, *Sur la détermination des premiers états d'un système de fermions dans le cas dégénéré*, *Nucl. Phys.* **8** (1958) 91.

[26] H. Feshbach, *A unified theory of nuclear reactions, 2*, *Ann. Phys. (NY)* **19** (1962) 287.

[27] P.J. Ellis and E. Osnes, *An introductory guide to effective operators in nuclei*, *Rev. Mod. Phys.* **49** (1977) 777.

[28] K. Suzuki and R. Okamoto, *General structure of effective interaction in degenerate perturbation theory*, *Prog. Theor. Phys.* **71** (1984) 1221.

[29] B. Brandow, *Linked-cluster expansions for the nuclear many-body problem*, *Rev. Mod. Phys.* **39** (1967) 771.

[30] R.F. Bishop, *An overview of coupled cluster theory and its applications in physics*, *Theor. Chim. Acta.* 80 (1991) 95.